Math 1512 - Exam 3 Study Guide

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April 5, 2023

Contents

Summary and Disclaimer

This is a study guide for the third exam for math 1512 at the University of New Mexico (Calculus I). The exam covers sections 3.9 and 4.1-4.5 of Stewart's Calculus. As such, this study guide is focused on that material. I assume that the student reading this study guide is familiar with the material previously covered in calculus 1, as well as the material from college algebra and trigonometry. If a you feel that you need to review this material, you can send me an email, or take a look at Paul's Online Math notes:

https://tutorial.math.lamar.edu/

If you are not in my class, I cannot guarantee how much these notes will help you. With that said, if your TA or instructor has shared these with you, then you will most likely get some use out of them.

Methods and Techniques

The focus of this exam is on basic integration. In particular, we focus on antiderivatives, Riemann sums, the Fundamental Theorem of Calculus (parts 1 and 2), and the substitution rule. This means that to really master the material, you need to have a solid understanding of derivatives.

Our first method is antidifferentiation, which tells us how to find the original function if we are given $f'(x)$. Because there isn't much to say about it (it is something you just need to be comfortable with because of your experience finding derivatives), we won't talk about it in great detail, but we will mention some things about finding antiderivatives in the second on indefinite integrals.

We will mention the Fundamental Theorem of Calculus, and we will discuss each part separately.

Fundamental Theorem of Calculus Part 1

For any number a (which doesn't depend on x), we have that

$$\left(\int_a^x f(t)\ dt\right)' = f(x).$$

The second part of the fundamental theorem of calculus is more useful in practice, and will be more important in the long run:

Fundamental Theorem of Calculus Part 2

If $F(x)$ denotes the antiderivative of $f(x)$ (that is, $F'(x) = f(x)$), then

$$\int_a^b f(x)\ dx = F(a) - F(b).$$

Next, we discuss indefinite integrals. These are effectively just antiderivatives, and are a more convenient way of writing them rather than "$F(x)$".

Indefinite Integrals

If $F(x)$ is the antiderivative of $f(x)$, we denote

$$F(x) = \int f(x)\ dx.$$

It is important to note that since $F(x)$ isn't unique (you can change it by adding any constant you would like), we always add a "+C" to the antiderivative, representing this constant.

Finally, we have the substitution rule.

Substitution Rule

In order to evaluate

$$\int f(x)\ dx$$

by substitution, we first choose our function u, and rewrite the integral in terms of u, say, $f(x) = g(u)$. Then we have

$$\int g(u)\ dx$$

To turn dx into du, we take $\frac{du}{dx} = f'(x)$, and rearrange. This gives us $du = f'(x)\ dx$.

The process is rather tricky, so make sure you work through a few examples to ensure that you understand it.

Worked Examples

We will now work through some examples. We start by taking some antiderivatives.

Example: If $f'(x) = x^2 + \cos(x) - \sin(x) + 3\sec(x)\tan(x)$, find $f(x)$.
We begin by splitting this apart on addition. The antiderivative of x^2 is $\frac{x^3}{3}$, the antiderivative of $\cos(x)$ is $\sin(x)$, the antiderivative of $-\sin(x)$ is $\cos(x)$, and the antiderivative of $3\sec(x)\tan(x)$ is $3\sec(x)$. So,

$$f(x) = \frac{x^3}{3} + \sin(x) + \cos(x) + 3\sec(x) + C.$$

Next, we should work a classic problem using the fundamental theorem of calculus.

Example: Find the derivative of

$$\int_0^{x^2} \frac{\sin(t)}{t}\, dt.$$

This problem is a classic because one cannot find a formula for the antiderivative of $\frac{\sin(x)}{x}$. Instead, we need to be clever. If we just write $F(x)$ for the antiderivative of $\frac{\sin(x)}{x}$, then we have by the fundamental theorem of calculus (part 2) that

$$\int_0^{x^2} \frac{\sin(t)}{t}\, dt = F(x^2) - F(0).$$

So, taking the derivative gives us that

$$\frac{d}{dx}\int_0^{x^2} \frac{\sin(t)}{t}\, dt = \frac{d}{dx}(F(x^2) - F(0)).$$

Then, we note that $F(0)$ is just a number, so its derivative is zero. And by the chain rule, $(F(x^2))' = F'(x^2)(x^2)' = 2xF'(x^2)$. And we know that $F'(x) = \frac{\sin(x)}{x}$. Thus, we have that

$$\frac{d}{dx}\int_0^{x^2} \frac{\sin(t)}{t}\, dt = 2x\frac{\sin(x^2)}{x^2} = \frac{2\sin(x^2)}{x}.$$

Practice Problems

These practice problems are separate from the unsolved problems. They should be used to make sure that you are confident with the material, and are of approximately the same level of difficulty as the unsolved questions. They also include worked solutions, unlike the unsolved questions section.

1. Find $\frac{d}{dx}\int_0^{x^3-1} e^{-t^2}\ dt$.

2. Find $\int \sec(x)\tan(x) + \sec^2(x) - x^3 - 5x - x^{\frac{1}{2}}\ dx$.

If it is requested, I will add more practice problems.

Practice Problem Solutions

1. Find $\frac{d}{dx}\int_0^{x^3-1} e^{-t^2}\ dt$.

Solution: By the fundamental theorem of calculus, if we write $F(x)$ for the antiderivative of e^{-t^2}, then

$$F(x^3-1) - F(0) = \int_0^{x^3-1} e^{-t^2}\ dt.$$

So, taking the derivative gives us that

$$\frac{d}{dx}\int_0^{x^3-1} e^{-t^2}\ dt = 3x^2 F'(x^3-1) = 3x^2 e^{-(x^3-1)^2}.$$

2. Find $\int \sec(x)\tan(x) + \sec^2(x) - x^3 - 5x - x^{\frac{1}{2}}\ dx$.

Solution: Taking the antiderivative of each part (splitting on addition), we have that the integral is equal to

$$\sec(x) + \tan(x) - \frac{x^4}{4} - \frac{5x^2}{2} - \frac{3x^{\frac{3}{2}}}{2} + C.$$

Unsolved Questions

Here is a list of 20 unsolved questions which I feel are of similar difficulty to what might be asked of you on an exam.

1. Evaluate
$$\int \cos(2x) + 3x^2 + \sec(x)\tan(x)\ dx$$

2. Evaluate
$$\int \tan^2(x) + x^2 - \sqrt{5x} + 3\ dx$$

3. Evaluate
$$\int 3\square^2 - \square^{\frac{2}{3}} - 3\ d\square$$

4. Evaluate
$$\int \sin^2(t) + \cos(t)(1 + \cos(t))\ dt$$

5. Evaluate
$$\int x^3 - 4x^{\frac{2}{5}} + 2x^{-5} + \sin(x)\ dx$$

6. Evaluate
$$\int t^5 - \cos(t) - t^2\sin(15t^3) - 5t^2\ dt$$

7. Evaluate
$$\int \cos(x) - \sin(x) + x\sin(x^2)\ dx$$

8. Evaluate
$$\int_{-3}^{3} \sqrt{9 - s^2}\ ds$$

9. Evaluate
$$\int_{-4}^{10} |x - 1|\ dx$$

10. Evaluate
$$\int_{-13}^{1} |5t - 1|\ dt$$

11. Evaluate
$$\int_{-\pi}^{\pi} \sin(2t) - \cos(2t)\ dt$$

12. Evaluate
$$\int_{3}^{5} |x| + |x^2| + |5x| - |x^2 + 1|\ dx$$

13. Evaluate

$$\int_{-1}^{3} t^3 - 3t \ dt$$

14. Evaluate

$$\int_{\frac{\pi}{2}}^{\pi} \sin(4s) + \cos(4s) \ ds$$

15. Evaluate

$$\int_{0}^{x} s^2 - 1 \ ds$$

16. Evaluate

$$\int_{-\pi^2-1}^{\pi^2+1} \sin(4x)\cos(4x) \ dx$$

17. Find the derivative (with respect to x) of

$$\int_{x^2}^{2} \frac{\sin(t)}{t} \ dt$$

18. Find the derivative (with respect to x) of

$$\int_{0}^{3x+1} e^{t^2} \ dt$$

19. Find the derivative (with respect to t) of

$$\int_{0}^{x} \cos(5t) - 3\sin(t) + 5t^4 - t^{-1} \ dt$$

20. Find the derivative (with respect to t) of

$$\int_{0}^{x} \sqrt{t^2 - 1} \ dt$$